

## Interrogation of Silent Features, Creative Reasoning and Function Sense for Graphing Functions

Muad M. Shiyab

Faculty of Education, Yarmouk University, Jordan.

Received: 22/4/2019

Revised: 22/7/2019

Accepted: 19/4/2020

Published: 1/12/2020

Citation: Shiyab , M. M.(2020) .  
Interrogation of Silent Features,  
Creative Reasoning and Function  
Sense for Graphing Functions.  
*Dirasat: Educational Sciences*,  
47(4), 498–509. Retrieved from  
<https://dsr.ju.edu.jo/djournals/index.php/Edu/article/view/2521>



© 2020 DSR Publishers/ The University of Jordan.

This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY-NC) license <https://creativecommons.org/licenses/by-nc/4.0/>

### Abstract

Graphs and Graphing function are crucial aspects of understanding, yet this is troublesome to students. This research paper offers a quite distinct way of thinking to help ease such difficulty. It focuses on analyzing, connecting silent features of functions with the improvement of graphing skills and concepts of concavities, and increasing and decreasing phenomena. Two methods were explored, examined and discussed, namely arithmetic silent feature and reasoning, and asymptotic silent feature and reasoning. Data collected through interviews and observation were qualitatively analyzed. The impact of learning such methods on enhancing graphing skills and on understanding concavity was remarkable. Students enthusiastic involvement, and use of basic skills and previous knowledge helped them in the process of assimilating and accommodating concepts that were previously procedurally learned. So, their graphing skills were significantly enhanced. Based on the study results, the researcher recommends that such methods be part of teacher's specialized pedagogical content knowledge the teacher could take into consideration and add to his/her repertoire for effective teaching and better outcomes.

**Keywords:** Graphing functions, creative method for graphing, Arithmetic silent features, asymptotic silent feature of functions.

### استنتاج السمات الصامتة والتبرير الابداعي والتدوق الاقتراني في تطوير رسم الاقترانات

معاذ محمود الشيباب

جامعة اليرموك

#### ملخص

المنحنيات ورسم الاقترانات لهما اهمية كبيرة في الفهم الرياضي، ولكنهما تشكلان عائقا امام طلاب الرياضيات. هذا البحث يقدم طريقة فريدة في التفكير لمساعدة الطلاب للتغلب على هذه العوائق. تركز هذه الطريقة على تحليل وربط الخصائص الصامتة للاقتران مع مهارات الرسم ومفهوم التقعر و ظاهرة التزايد والتناقص. طريقتان تم التعرف على مدى فعاليتها وتم مناقشتهما: طريقة الخواص الصامتة للعمليات الحسابية بين مكونات الاقتران وتبريرها، و خاصية المحايديات الصامتة وتبريرها. جمعت البيانات من خلال المقابلات وطريقة الملاحظة وتم تحليل المشاهدات التي جمعت نوعيا. كان تأثير تعلم هاتين الطريقتين على تطوير مهارتي الرسم، وفهم مفهوم التقعر ملفتا للانتباه. مشاركة الطلبة بحماسة واستثمار المعرفة السابقة والمهارات الاساسية ساعدهم في عمليات التدويت (التمثيل) (assimilation) والاستيعاب (accommodation) للمفاهيم التي تم تعلمها سابقا بشكل اجرائي. أدى مرور الطلبة في هذه الخبرة الى تطوير مهاراتهم في رسم الاقترانات بشكل ملحوظ. بناءً على نتائج الدراسة، يوصي الباحث بان تكون هذه الاساليب جزءاً من معرفة المدرس في المحتوى البيداقوجي التي ينبغي اخذها بعين الاعتبار و اضافتها الى مخزونه لتحقيق تعليم فعال ومخرجات افضل.

الكلمات الدالة: رسم الاقترانات، طريقة رسم ابداعية، طريقة الحساب الصامتة، طريقة المحايديات الصامتة للاقترانات..

## Introduction

Connecting multiple representations is considered a major part of understanding mathematics and help provide reasoning and justification. The ability to represent mathematical processes, imagining how processes and actions may impact representations, anticipating and confirming the consequence of mathematical processes through modeling or graphing reflects understanding at a multidimensional level.

First, representation for mathematical phenomena such as a graph or any other form of representation reflects a certain type of understanding. For example, if the first derivative is negative on some interval and the second derivative is positive on the same interval, understanding this may be reflected by the ability to draw a decreasing and concave up figure, and also provides reasoning to describe the nature of the rate of change.

Second, making the connection between multiple representations and observing how a change in one representation can impact or cause change on another representation is another type of reasoning and understanding.

Third, imagining or seeing how one mathematical phenomenon embedded in another phenomenon is a form of understanding too. The ability to imagine or read a graph embedded in another graph is a form of coherence understanding and a form of connection (e.g. imagining the tangent lines for a curve and the ability to draw the graph that represents the tangent lines). Hirst suggested the skills that should be developed to enable learners to build conceptual non-superficial knowledge are creative in nature. Among these skills are:

1. Creative Imaginative skills
2. Creative thinking skills
3. Creative evaluative skills
4. Creative communicative skills (Hirst and White, 1998).

Creative thinking is defined by Facione (1998) as the kind of thinking that leads to new insights, novel approaches, fresh perspectives, and a whole new way of understanding and conceiving things (McGregor, 2007, p. 171). Creative thinking also was defined by Puccio and Murdock (2001) as creative thinking which is not just the generation of a novel idea, but also the multifaceted way in which it can be constructed and communicated (McGregor, 2007, p.168).

Building a mathematical concept and skill can't be done instantly, but it takes time, struggle, disposition, reflection, reasoning, management, and different ways of solving and thinking (Kelly, 2009; Glazer, 2001; Sriraman and English, 2010). Those skills and processes are supposed to be practiced besides, the mental operations involved in understanding. Those operations are Identification, Discrimination, Generalization, and Synthesis (Sierpiska, 1994). Almost infinitely many skills and operations are required for building conceptual understanding. Sapir defined the term concept as "a convenient capsule of thought that embraces thousands of distinct experiences and that is ready to take in thousands more"(Swan, 2001, p.152 ). Out of this scenario, it is not only not enough to say that teacher is not required to submit information, rather the major part of the teacher's task is to help students individually through multiple channels, such as social interaction, to make sense of the mathematical phenomena, and building more conceptual links and multiple perspectives (Swan,2001).

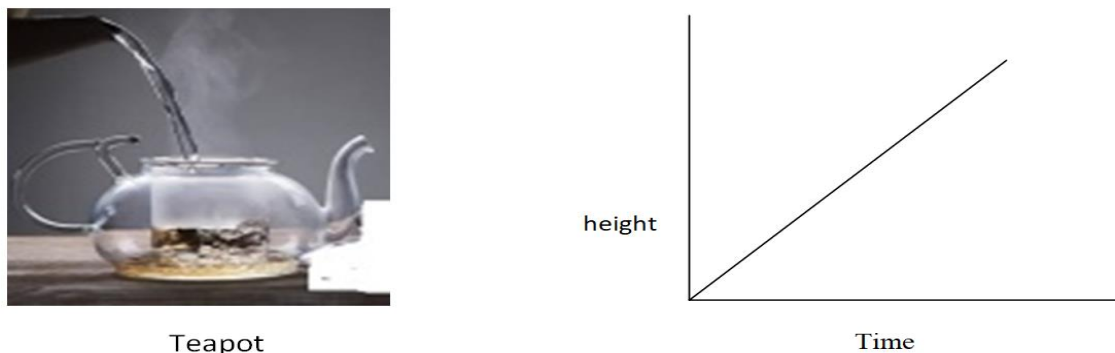
The conceptual links are forms of understanding, which is according to Piaget to produce a creative and productive learner; learner's understanding requires discovery, and reconstruction (deeper understanding); rediscovery will yield new conceptual links. This requires different ways of thinking and interacting with the surrounding environment, including mathematical phenomena and concepts. This kind of interaction allows assimilation, and accommodation, which in turn may yield to conceptual construction as a result of the absorption of new ideas. Moreover, absorption yields to alteration and change for the learner's cognitive structure. This change in the cognitive structure is due to the new links and new conceptual structure (Swan, 2001; Pritchard, 2009; Anderson and others, 1994). Research describes the cognitive change as 1) pervasive, 2) central, and 3) permanent, which are the three main conditions used to decide the occurrence of conceptual learning (Ben-Hur, 2006; Resnick, 1988).

Graphing functions are a phenomenon representing one of the difficult situations confronting mathematics-major students (Arnon, et al., 2014). One of the best undergraduate students in the mathematics program drew an increasing

straight line to represent the height of liquid being poured in a non-uniform teapot (Figure-1). This was the first incident that grabs the author’s attention to investigate it farther.

**Figure 1**

*Graph represents the height of the liquid inside the teapot*



**The Problem**

Over five years of teaching a course on methods of teaching mathematics students (pre-service teachers) at their fourth year in the undergraduate mathematics program, it has been found that they were facing the difficulty of graphing functions and modeling real life situations. Although they were able to deal with the differentiation procedures successfully, their disability to sketch a graph that reflects their work of differentiation, indicating that the problem could be due to several reasons. Arnon (Arnon, 2014) proposed that students’ inability to draw a graph is because they did not show thematization of the graphing Calculus schema. Yet, our hypothesis for such a problem is that, as suggested by Swan, salient, but silent is "failure to notice silent features of a situation"(P. 147). This failure may be hypothesized because teaching focuses on the procedural work; procedures of finding derivatives to determine increasing and decreasing intervals as well as the concavity intervals, without mulling over to reflect and interpret such signs by connecting the graph with the derivatives and the silent features of the underlined situation or function in order to develop a "function sense".

**Methodology**

This paper focuses on investing the role of silent features in helping the pre-service teacher to develop function sense and graphing skills. The mediation processes focus on reflection, discussion, and validation through the sense-making and use of technology. Student’s function sense and graphing skills were assessed through discussion and interviews. Data collected through discussion and interviews were qualitatively analyzed and discussed.

**Participants**

Twenty-three pre-service teachers (15 females and 8 males) participated in this study. Those students were enrolling in a ready- made-software course. They all had a bachelor's degree in mathematics, graduated from different schools in the country. The author, the teacher, of the course, provided 4 training sessions to discuss with the students how silent features could help them make sense of functions and enhance their graphing skills.

**Procedure**

In the first step, participants were given the functions  $f(x) = \frac{x^2-2}{x}$  and  $f(x) = \sec x$  to examine student’s ability and skills for graphing functions, students were not limited in time, rather they were given open time, and then their work was collected and evaluated.

The author focuses his method of teaching on silent features (Asymptotes) that we see and deal with but rarely invested as tools and indications to improve graphing skills and understanding concepts such as the rate of change, concavity, and increasing and decreasing phenomena. We also use critical points as indicators to guide us to decide the behaviors of a function as  $x$  values get close to the critical value, but we do not discuss or connect it explicitly with the rate of change.

Likewise, we view and treat asymptotes. Seldom asymptotes are unequivocally used to discuss concavity, or increasing and decreasing phenomena. So, these two methods focus on how to connect silent features; where silent feature refers to features in which its impact is not explicitly linked with, nor it is discussed through the traditional procedures that are used with concavity and graphing skills. The author instructed participants that they can pose questions, objections, and feedback freely. So, students' questions, objections, and feedback were discussed in an investigational manner while the author was explicating his method.

**Demonstration of the methods**

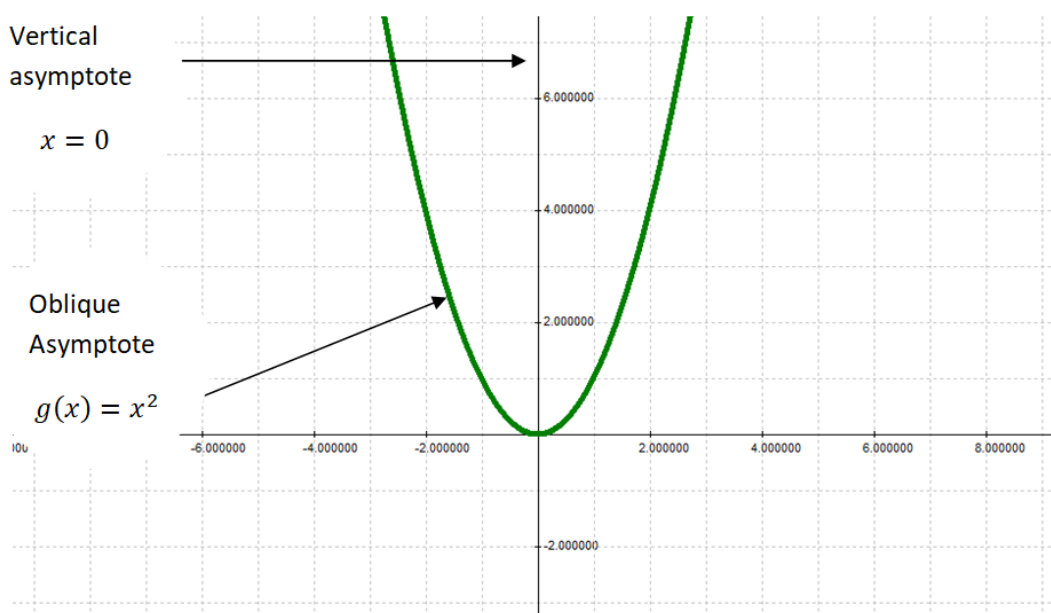
In the second step, the author of this research started the activity by explaining the underlying philosophy behind this method; which it was empowered by the suggestions of (Hirst, 1998) regarding the thinking skills.

In the third step, the teacher wrote on the board the first silent feature which is “The asymptote silent feature and reasoning”. The function  $f(x) = \frac{x^2-2}{x}$  was written on the board. The zeros of the denominator were determined ( $x = 0$ ). Students were asked about the impact of such a value on the graph. Then, the teacher discussed with them that the function  $f(x) = \frac{x^2-2}{x}$  has a vertical asymptote  $x = 0$ ,  $g(x) = x^2$  is oblique asymptotes (Figure-2). This means that as  $x$  goes to  $\infty$  the graph of  $f(x)$  has to be closed to the graph of  $g(x) = x^2$ . Moreover, as  $x$ -values approach 0 from the right side, the graph of  $f(x)$  will approach  $(-\infty)$ . Since the sign of  $f(x)$  is negative as  $x$  goes to zero, given that the  $y - axis$  is vertical asymptotes; this implies that as  $x$  values get close to 0, the graph of  $f(x)$  will start to decrease rapidly as it is approaching  $-\infty$ . Hence, the graph will have to concave down more and more. Because, in a short interval  $(0,1)$  the values of  $f$  will have to approach  $-\infty$  as we move toward 0 from the right side. This reflects the fact that the rate of change is decreasing at an increasing rate of change as  $x$  approaches zero. Hence, the graph from the right side can be connected to the  $x$  values and  $y$  values like this: as the values of  $x$  goes to zero the values of  $y$  will have to rapidly approach  $(-\infty)$ . Approaching rapidly  $(-\infty)$  could be interpreted as decreasing in an increasing rate of change. This interpretation should not be confused with the fact that the function  $f(x) = \frac{x^2-2}{x}$  is increasing on the interval  $(0, \infty)$ , and it concaves down on the interval  $(0, \sqrt[3]{2})$ .

On the other hand, as  $x$  goes to  $\infty$ , values of the function  $f$  will get closer and closer to the values of  $g$ , and this will produce a concave up-ward; since  $g(x) = x^2$  is concave up.

**Figure 2**

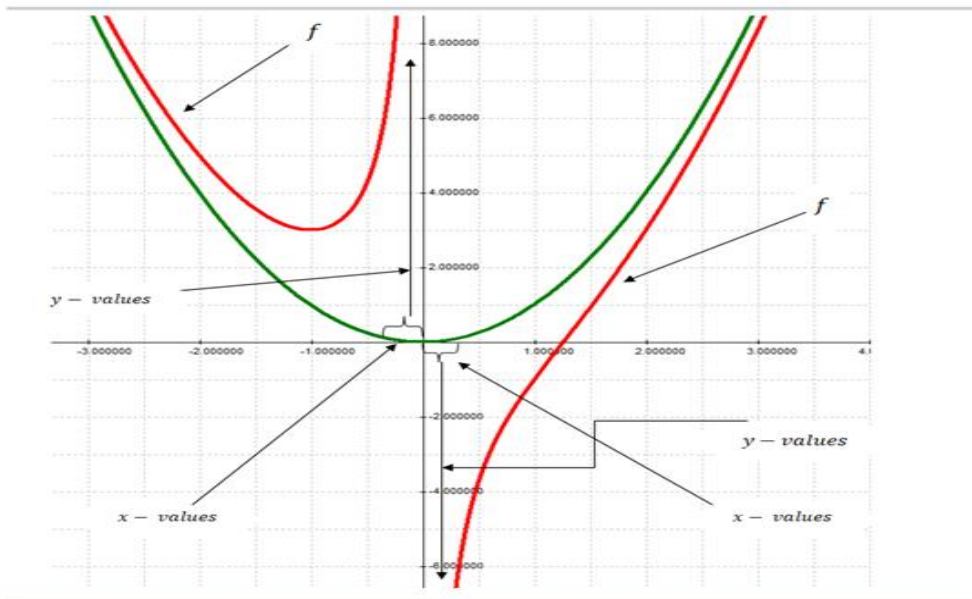
*The graph of the oblique asymptotes*



The other (left) side of the curve (Figure 3) has a great potential for fostering understanding for "increasing in an increasing rate of change, which means the graph has to be concave up". In such a short interval(-1, 0), the values of  $f$  shall increase rapidly to approach  $\infty$ . Hence, the graph of  $f$  must be concave up, and it is impossible to be concave down, otherwise, the graph of  $f$  will intersect the  $y - axis$  (asymptotes).

Figure 3

Graph of  $x^2$ , and  $\frac{x^3-2}{x}$



In the fourth step, together we reflected on how both asymptotes (vertical and oblique) are used to understand concavity, and it lends a hand in enhancing students' imagination to the curve as a whole.

The fifth step, the function  $f(x) = \sec x$ , and the first and second derivative Table 1, were used to introduce the arithmetic silent feature and its reasoning. The teacher wrote on the board "Arithmetic silent feature and its reasoning"

Table 1

The function  $\sec x$ , the first and second derivative

<p>Trigonometric problem</p> <p><math>f(x) = \sec x</math></p> <p><math>f'(x) = \sec x \tan x</math></p> <p><math>f''(x) = \sec x(1 + 2\tan^2 x)</math> or</p> <p><math>f''(x) = \sec x(\tan^2 x + \sec^2 x)</math></p>
---

The teacher then asked what is the  $\sec x$ . The majority of the participants respond by saying it is  $\frac{1}{\cos x}$ . Reflecting at the  $f(x) = \cos(x)$  graph in figure-4. The range of  $f$  is  $[-1, 1]$ , and the zeros of the  $\cos(x)$  are  $(2n + 1)(\pi/2); n \in \mathbb{Z}$ . Moreover, the  $\sec x = \frac{1}{\cos x}$ , so the  $\sec x$  arithmetically is the result of continuous division processes;  $y = 1$  divided by all the values in  $[-1, 1]$ . The values in Table 3 are just a sample to clarify why the function  $f(x) = \sec x$  is an increasing function when it is increasing. The value of the  $\cosine$  at  $x = 0$  is 1, so,  $\sec(x = 0) = \frac{1}{1} = 1$ , at  $x = \frac{\pi}{2}$  the value of the  $\cos x = \frac{1}{2}$ , and hence the value of the  $\secant$  is  $\frac{1}{\frac{1}{2}} = 2$ , and the value of the  $\secant$  at  $\cos x = \frac{1}{3}$  is  $\frac{1}{\frac{1}{3}} = 3$ , and so on for the rest of the  $\cosine$  values between 1 and 0 in the  $x$ -interval of  $[0, 1]$ . At the largest value of the  $\cosine$  will be the smallest value for the  $\secant$  as long as the  $\cosine$  is positive. The justification is

that "1 is divided by decreasing values for the *cosine* (between 1 and 0) as we move right on the *x - axis* surely will get larger values for the *secant*".

**Table 2**

*Values of the secx given some values for the cosx*

<i>cosx</i>	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
$secx = \frac{1}{cosx}$	1	2	3	4	5	6	7	8

Data in Table 3 shows that the values of the function  $f(x) = secx$  increases at an increasing rate of change. Observe that for every unit change in *x*, the values of the *secant* function increases more than it was increased in the previous one. Table-3 shows the first differences between the values of the secant. The first difference is an increasing sequence, so we do not add the same amount each time. This explains the fact that *secant* increases at an increasing rate of change. Moreover, at every single value of the set  $(2n + 1)(\pi/2)$

the denominator of the  $secx = \frac{1}{cosx}$  will equal zero, hence the *secant* values will approach  $\infty$  as *x* approaches 0. Notice that in a short interval on the *x - axis*, the values of the *cosine* will equal zero, this leads the secant values to approach  $\infty$ , and will cause a rapid increase in the amount of change leading to a rapid increase in the secant values. This explains the concavity.

**Table 3**

*Values of the secx given some values for the cosx*

<i>Cosx</i>	1	$\frac{9}{10}$	$\frac{8}{10}$	$\frac{7}{10}$	$\frac{6}{10}$	$\frac{5}{10}$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$
$secx = \frac{1}{cosx}$	1	$\frac{10}{9}$	$\frac{10}{8}$	$\frac{10}{7}$	$\frac{10}{6}$	$\frac{10}{5}$	$\frac{10}{4}$	$\frac{10}{3}$	5	10

**Table 4**

*Values for secx, and first differences between every two consecutive values*

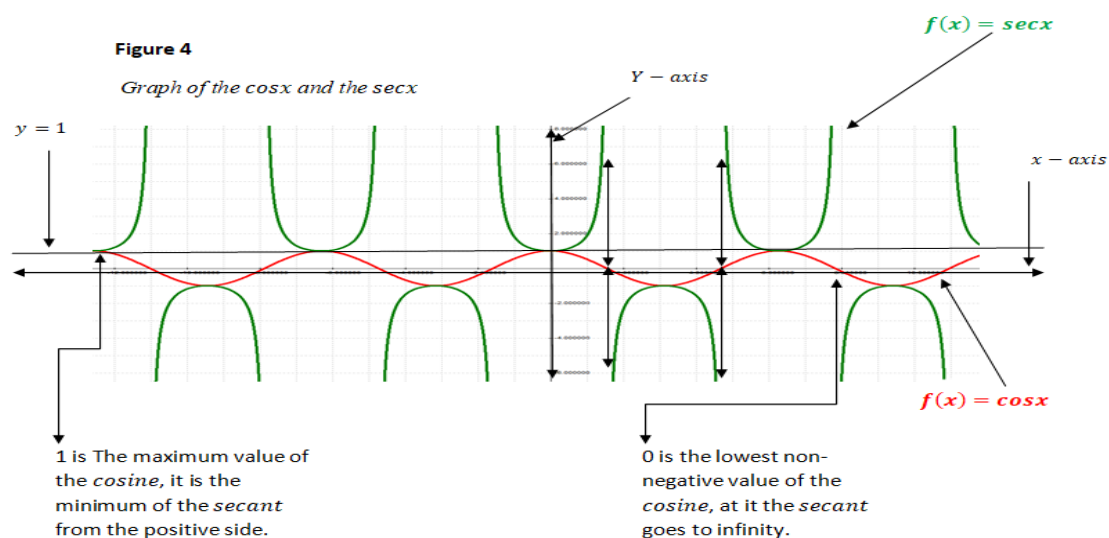
<i>secx</i>	1	$\frac{10}{9}$	$\frac{10}{8}$	$\frac{10}{7}$	$\frac{10}{6}$	$\frac{10}{5}$	$\frac{10}{4}$	$\frac{10}{3}$	$\frac{10}{2}$	10
1 <sup>st</sup> difference		$\frac{1}{9}$	$\frac{10}{72}$	$\frac{10}{56}$	$\frac{10}{42}$	$\frac{10}{30}$	$\frac{10}{20}$	$\frac{10}{12}$	$\frac{10}{6}$	$\frac{10}{2}$

Similar reasoning for the negative part of the cosine can help understand the relationship between the *cosine* and the *secant* from the arithmetical and geometrical view shown in Figure 4.

After this method was explained, students individually and in pairs were given functions to be graphed by applying this method. Two pairs of functions were given to students to practice.

1.  $f(x) = \frac{x^2-3}{x-4}$ ; and  $g(x) = \frac{x^2+4}{x}$
2.  $f(x) = \csc x$ ; and  $g(x) = \tan x$

Students were instructed to work individually first, and then to work in pairs. Nevertheless, they were allowed to move and discuss the way they wished. Moreover, they were asked to try to reflect on the nature of those functions, try to make sense of them, and provide reasoning for the curves they produce.



### Result

Before participants were shown both methods of graphing when they were given the function  $f(x) = \frac{x^2-2}{x}$ , many of them started looking at each other silently with blurring eyes reflecting despair and hopelessness trying to seek help from each other. Soon they stop working; it did not take them more than five minutes to announce that they can't graph such function. For some of them even though they were able to determine the signs of the first and second derivatives, they could not translate their finding into a graph. Some others returned their papers blank and express that it is too difficult to graph such functions. Whereas, some of them draw meaningless semi graphs without any reasoning.

After the demonstration of the methods as soon as they were given the first pairs of functions, the most noticeable behavior students shown was that they started to analyze the components of the function to have a deeper understanding of the function. They started to determine where the function does not exist and why; try to interpret the zeros of the denominator. They began to guess how the graph may look like. Students began to make sketches, trying to make sense by going back and forth between the algebraic and their geometric sketches.

The following graphs shown in figure5 and 6 are just a sample of student's drawing, They did them amazingly in a short time, they reasoned, and validated their drawing through discussion by using arithmetic and asymptotic reasoning. They even started to predict the sign of the derivatives and validated its sign through their drawing. One of the comments a student has made as she was trying to understand  $f(x) = \frac{x^2-3}{x-4}$  Figure 6 was that:

"at the infinity, the graph is almost linear since it is  $x$  of the fifth degree over  $x$  of the fourth degree,  $y = x$  is an asymptote which means that there will be no increase in the amount of change and that is why it is not going to be concave any more".

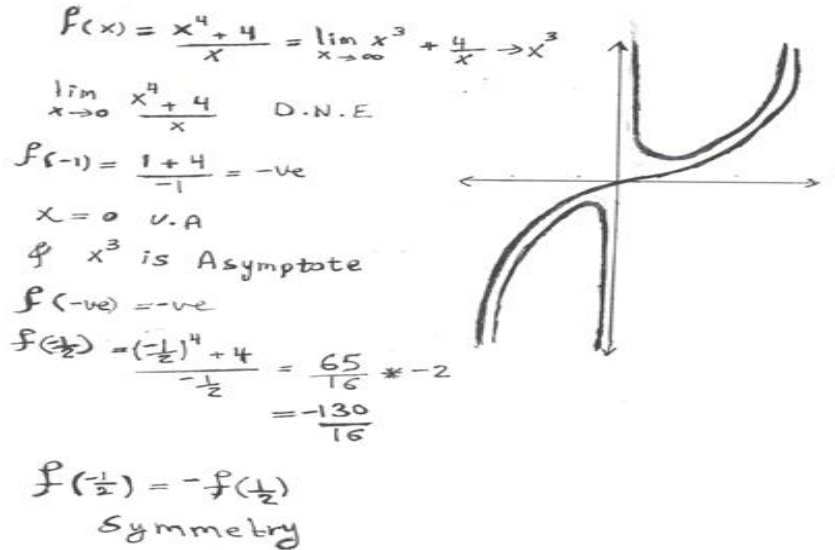
Another student reasoning for his drawing for the function  $f(x) = \frac{x^2+4}{x}$  Figure 5 was that:

"It is closed to the graph of the function  $x$  cubed, and the  $y$ -axis is an asymptote. So, but as I get close to 0 from the positive  $x$ -axis, the values of the function are positive. So it has to go to positive infinity, let see from the left side if I plug negative half to the power 4 will be positive plus 4, so it is positive divided by negative, the result is negative, it goes to

minus infinity. So the graph must be bounded by two asymptotes".

Figure 5

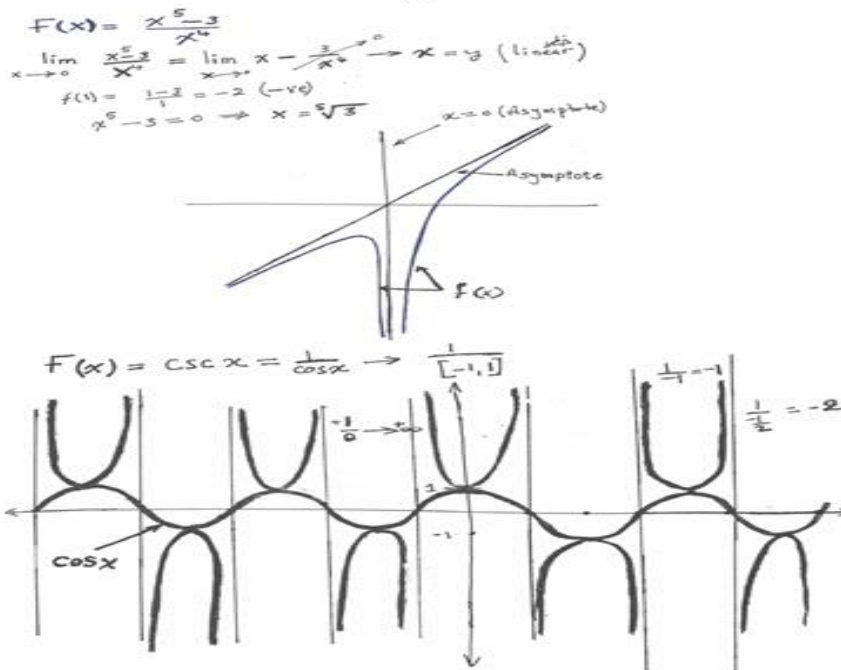
Student drawing for the function  $\frac{x^4+4}{x}$



When they tried to deal with  $\tan x$ , students viewed it as  $\frac{\sin x}{\cos x}$ , they graph the  $\sin x$  and  $\cos x$  separately. For accuracy, they were allowed to use software to graph both functions; the  $\sin x$  and  $\cos x$ , as in figure-7, but they were not allowed to graph the  $\tan x$  unless they have tried to graph it on their own and provide reasoning. They started to apply the division processes. Students in one of the groups were dividing verbally by just looking at the figures. "they started the division processes at  $x = 0$ , by dividing the.

Figure 6

Student drawing for the function  $\frac{x^5-3}{x^4}$  and  $\csc x$





Value of the **sine (1)** by the value of **cosine (0)**, and they divided both functions values at the intersection and they continued to divide until they got to the point where they had to divide the value of the **sine (1)** by the value of the **cosine (0)**. They found out it is  $\infty$  and they identified the angle to be  $\pi$ . One of the students drew the graph shown in figure 8. The student's reasoning was

"the values of the **sine** are increasing in the interval zero and ninety degree and the values of the **cosine** is decreasing in the same interval since we are dividing increasing values by decreasing values, hence the values of the **tan** will increase".

Another student responded by saying :

"yes. it has to increase rapidly because the values of the **tan** have to reach  $\infty$  as  $x$  values reach ninety degrees, and it can't intersect this line(referring to the asymptote at the ninety degrees). So, it has to be concave up".

She uses the asymptote to decide and validate the concavity.

Table 5

Values for the  $\sin x$ ,  $\cos x$ , and  $\tan x$

$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	+999
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	+0	-0
$\tan x = \frac{\sin x}{\cos x}$	0	$\frac{1}{\sqrt{3}}$ $\cong .57$	1	$\frac{\sqrt{3}}{1}$ $\cong 1.73$	$+\infty$	$-\infty$

Other students built tables like the one shown in table 8, and plotted those points. What guides them the most to realize that the curve has to be concave up in the interval  $[0, \frac{\pi}{2}]$  is the cosine value (0) and sine value(1), which causes the  $\tan(x)$  to go to infinity.

Figure 7

Graph of the  $\sin x$  and  $\cos x$

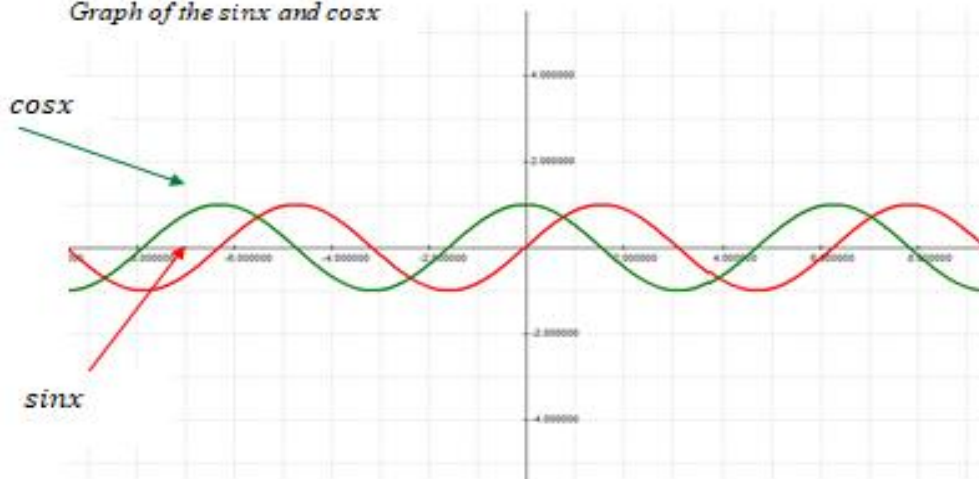
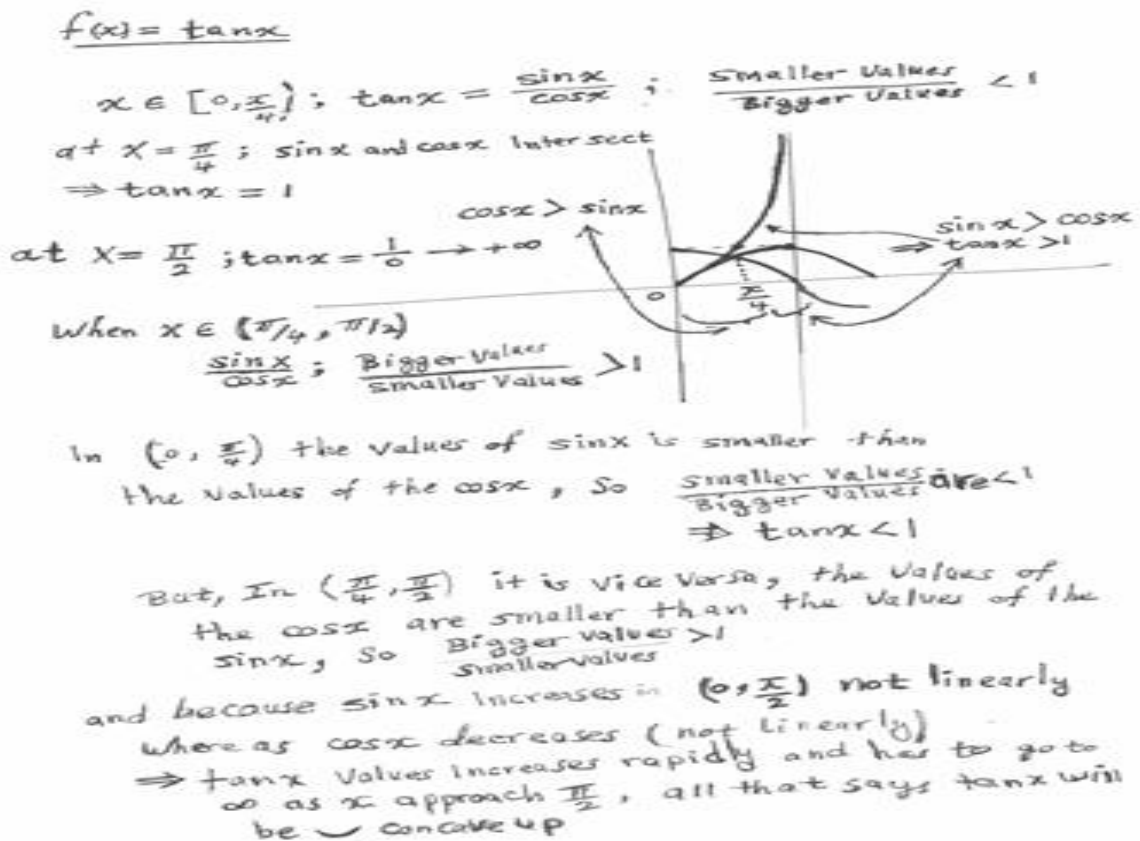


Figure 8

Students drawing for  $\sin x, \cos x, \tan x$ 

### Analysis of the Findings

#### Pre-observation analysis

Before they learned the two methods, students were given a function to draw its graph, some students start by finding the first and second derivatives, and the intervals on which the functions had positive derivatives and the intervals on which the function had a negative derivative. Some succeeded in doing so, and some fail. However, those who were able to find the signs of the derivatives fail to convert their findings into a graph. And hence for them, there was no other option to deal with the function. Their failure could reflect their disability to interpret and convert the sign of the derivatives into action. Converting their findings into action and process required appropriate interpretation and understanding of the links between the graph of a function and its rate of change represented by the derivatives. Moreover, even those very few of the participants who tried to draw a sketch, and even though they did not do it correctly, they were not able to provide reasoning or defend their trial, except they kept saying "according to the sign of the derivatives". When they were asked if they can imagine how the graph may behave, they stared at each other as they were shrugged their shoulder astonishingly and said do you expect us to do so. This reflects their inflexibility and despondency.

When they tried  $\sec(x) = \frac{1}{\cos x}$  as they found the first derivative ( $\tan x \sec x$ ), they were stuck in finding the derivative sign and stop trying. When they were asked to try more options or different methods, they were amazingly deplored of my suggestion, as if they were saying what else out there to try beside derivatives. This reflects that their understanding was at the procedural level; when they were stuck, they stop because they had no other choice to try.

When it comes to graphing functions, derivatives are used as a tool and indication to decide where the function is increasing or decreasing and to decide the concavity too. It helps also to locate critical points. But the derivative by itself does not tell about the nature of the function, nor it provides the reasoning behind the positive derivatives or negatives.

### ***Post-observation analysis***

The result of learning the two methods is that participants were provided with the opportunity to reflect on the essence of the function and its nature. For example, one form of the  $\sec(x)$  is  $\frac{1}{\cos(x)}$ , so actually, it is the division of one function into another. It gives students an opportunity to imagine and build a very clear understanding of what is happening and helps them to make sense and interrogate reasoning from the components of the function and its relationships among each other. This alternative way of thinking also helps to develop a student's ability and flexibility to understand and deal with complex situations.

This way of dealing with functions gives learners an opportunity to build schemes, which involves not only the graph of the function and its characteristics, but it also involves relationships and dynamic relationships among the sub-functions. It helps the learner to predict the behavior of similar functions. The processes of performing the operations help enhance student's skills of re-applying them as well as enhance students' understanding of the underlying relationships among the components of the function. Moreover, students can make sense of the impact of the operations among the components of the functions.

In this part, students were freely discussing the essence of the functions as well as the process involved to produce a graph. They used what they already know, such as arithmetic and division processes of numbers; they extended the arithmetic processes to algebraic processes by extending the division of discrete to the division of continuous functions. They surprisingly grasped the method and started to divide geometrically without using numbers and calculators. They were dividing as they were tracing both functions in the nominator and denominator geometrically. This thinking process led to the assimilation of the concept of derivatives and its indications.

They validated their graph and solutions through analysis and connecting their observation with the interpretation of the first and second derivatives; this process is considered an accommodation process, where they interweave assimilation and accommodation through conjecturing the sign of the derivative (without deriving) and reasoning. Testing and adjusting with explications to each other through the function sense-making, something we all strive to achieve.

Their covert and overt engagement and seriousness were most attractive; they were going back and forth between using the board, computers, paper, and pencil to draw and explain their points of view. Listening to each other and using each other ideas for elaboration and validation. For the author, this sophisticated and constructive level of interaction is even far more important than just learning abstract math, storing facts and executing procedures. It helps students enhance positive disposition, motivation, and attitude toward mathematics, and learning mathematics. Student's remarks such as the statements below are one example

*"I never thought it is as simple as arithmetic.. aha now I understand what is the positive or negative sign of the first and second derivative means"*

reflect confirmation of understanding the interpretation of the derivatives through connecting the algebraic result with the nature of the functions and its constituents through reasoning.

These two methods provide different ways for investigation and exploration for connections and links among different concepts. Indeed, it unfolds the implicit and explicit meaning to the rate of change, derivatives, increasing and decreasing, and concavity to serve the enhancement of graphing skills. They start with the bottom-up approach through experimenting and sense making and ended up using the top-down approach through justification, deducing and interpretation of derivatives and other theories. Making sense of function characteristics through simple arithmetic help in structural cognitive change which it led to the enhancement of their graphing skills by converting their sense into action. Understanding graphing skills and understanding concavity and the ability to apply them properly in one sense means assimilating them into an appropriate schema. Students employ operational and conceptual schemata to develop such understanding.

### **Summary**

The recapitulation of this research experience is that the processes of description, analysis, and synthesis of concepts, sub-functions, and operations that students experience through what I called general and specific coordination and understanding of all the components of the function was rich, excited and valuable. For example, for graphing the function

$f(x) = \frac{x^2+4}{x^2}$  students visualize and discuss the following:  $x^4$ ,  $x^4 + 4$ ,  $x$ ,  $x^3$ ,  $\frac{x^2+4}{x^2}$ , asymptotes, division process, concavity, increasing and decreasing, tangent lines and all the dynamic interrelationships among all these components. If this is compared with traditional derivation and graphing in which the major focus is on performing the process of derivation abstractly, then the difference is obvious and the chasm is so wide.

Overcoming the inertia that inclines students to freeze when it comes to drawing did not take place in isolation from the social culture in which students were invited and encouraged to try to perform investigational activities, and invited to take on a more active role. This positive end was possible mainly due to the exploration and experimentation of silent features for mathematics concepts. Something we all as educators strive to accomplish.

### Conclusion and Recommendation

The interrogation of silent feature showed that it is an effective approach in helping students use their basic knowledge and basic component of a function, and invest basic operations among sub-functions, and asymptotes to understand and decide the behaviors of functions. This is a novel method that helps deepen students' understanding and develop their creative and flexible skills to deal with complex functions and phenomena.

Moreover, the classroom culture was designed through the lenses of considering teaching mathematics as a system that coordinates several domains; domains such as equity and accessibility; nature of the classroom task; teacher as well as students role; mathematical tools and digital technology (Schoen, 2003; Lester, 2003), to foster for the components of metaknowledge for intentional learning (Resnick, 1989).

I propose such methods to be part of Teacher's specialized pedagogical content knowledge that the teacher could take into consideration and

add it to his/her repertoire for effective teaching and better outcomes.

*Finally, I hope that this paper provides some sort of analysis that operates as a catalyst for reflection on the subject under study and as a forum debate for future works.*

### References

- Anderson, R. D. (1994). *Issues of curriculum reform in science, mathematics, and higher order thinking across the disciplines*. US Department of Education, Office of Educational Research and Improvement, Office of Research.
- Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). APOS theory: A framework for research and curriculum development in mathematics education.
- Ben-Hur, M. (2006). *Concept-rich mathematics instruction: Building a strong foundation for reasoning and problem solving*. ASCD.
- Glazer, E. (2001). *Using internet primary sources to teach critical thinking skills in mathematics*. Greenwood Publishing Group.
- Hirst, P.H. and White, P. (2001). *Philosophy of Education: Major Themes in the analytic tradition Philosophy and Education*. New York: Routledge in Taylor & Francis e-library.
- Kelly, A. V. (2009). *The curriculum: Theory and practice*. Sage.
- Lester, F. K. (2003). *Teaching mathematics through problem solving: Prekindergarten-grade 6*. National Council of Teachers of Mathematics.
- McGregor, D. (2007). *Developing thinking; developing learning*. McGraw-Hill Education (UK).
- Pritchard, A. (2009). *Ways of learning: learning theories and learning styles in the classroom*: David Fulton Publish.
- Resnick, B., L., (1989). *Knowing Learning and instruction*. New Jersey, Hillsdale: Lawrence Erlbaum Associates, Inc.
- Schoen, H. L. (2003). *Teaching mathematics through problem solving: Grades 6-12*. National Council of Teachers of Mathematics.
- Sierpinski, A., (1994). *Understanding in Mathematics*. London: The Falmer Press.
- Sriraman, B., & English, L. D. (Eds.). (2010). *Theories of mathematics education: Seeking new frontiers* (pp. 309-331). New York: Springer.
- Swan, M. (2001). 10 Dealing with misconceptions in mathematics. *Issues in mathematics teaching*, 147.